

Light Sheets and the Covariant Entropy Conjecture

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Abstract

We examine the holography bound suggested by Bousso in his covariant entropy conjecture, and argue that it is violated because his notion of light sheet is too generous. We suggest its replacement by a weaker bound.

1 Introduction

Since the original papers of 't Hooft and Susskind [1, 2], much work has been done on the notion of holography; see Smolin [3] and references therein for a review. In particular, Fischler and Susskind [4] suggested a holography bound in terms of null surfaces in the context of cosmological solutions. Bousso [5, 5, 7] has since stated and made use of a conjectured holographic entropy bound which generalises the work of Fischler and Susskind by relating the area of a two-surface to the entropy contained on a section of null hypersurface which ends on that two-surface in general space-times. This work also builds on earlier work of Bekenstein [8] relating the entropy of a region to its surface area under certain constraints.

We briefly recall the Fischler-Susskind-Bousso (FSB) bound.

Let Σ be a spacelike two-surface in a space-time, \mathcal{M} . We wish to give a bound in terms of A , the area of Σ , for the entropy contained within Σ . There is an ambiguity here, since many three-surfaces in space-time span the surface Σ . One approach to removing this ambiguity is to obtain a natural candidate for the region bounded by Σ whose entropy is to be considered. So consider the four orthogonal null (geodesic) congruences to Σ ; then a light sheet, L , is given by an orthogonal null congruence which has negative expansion everywhere on Σ , each null geodesic being extended to the first point at which the expansion vanishes (i.e. the first point along the null geodesic which is conjugate

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to Σ). Bousso's conjecture is, then, that the entropy contained on L , denoted by $S(L)$, satisfies the inequality

$$S(L) \leq \frac{A}{4}.$$

This inequality is the FSB bound, and the statement that it holds in general space-times is the covariant entropy conjecture. It is supported by various examples in the context of standard cosmological solutions. However, the explicit examples provided have relied heavily on Σ having spherical symmetry. It is the aim of this paper to argue that in the more general situation where spherical symmetry of Σ is dropped, the FSB bound is violated, and that in general, the light sheet should only include that portion of the null congruence up to the first conjugate point or point of self-intersection on each null geodesic.

It should be noted that this argument does not attempt to invalidate the programme being carried out by Bousso, only to establish that the inequalities used are too strong, and should be replaced by weaker ones when appropriate.

2 Ellipsoidal examples

The motivation for the following section is the observation that in Bousso's examination of the covariant entropy conjecture, situations were noted where the inequality was saturated, even in the case where Σ has spherical symmetry. For such surfaces, the light sheet is cut off by conjugate points before any self-intersection occurs. This suggests that it might be possible to find a counter-example by considering a light sheet at least part of which extends far beyond the first self-intersection before the first conjugate point occurs. To this end, we will consider first the situation for a field of matter on a Minkowskian background, and then extend this to the consideration of full solutions of the Einstein equations.

So we begin by considering the ellipsoid, Σ , given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

in the $\{t = 0\}$ slice of Minkowski space, with the usual coordinates (t, x, y, z) . Then take as light sheet, L , the relevant portion of the future pointing inward directed orthogonal null congruence to Σ . It is clear that as b approaches 0, the ellipsoid approaches (two copies of) the disk $x^2 + y^2 \leq a^2$, while the associated light sheet approaches two pencils of null geodesics.

By taking b sufficiently small, the area, A , of Σ approaches $2\pi a^2$ arbitrarily closely, while the extent of L increases without bound. It then follows that if Minkowski space is the background on which lies a stationary matter field whose entropy density is constant, then by taking b sufficiently small, the entropy passing through L may be made unboundedly large, and so for some $b > 0$, the FSB bound must be violated.

This tells us that the covariant entropy conjecture fails for stationary matter on a flat background (which is a first approximation to the cosmological case), but of course this

does not necessarily imply that the conjecture itself must be violated; it merely suggests that concern is justified. We proceed in steps of decreasing idealisation.

To this end, we next consider a FLRW space-time with constant co-moving entropy density, as in Tavakol and Ellis [9]. In this case, we consider an ellipsoid in a surface of constant time. By taking the ellipsoid sufficiently small, we obtain a situation which is well-modelled by flat space-time again. Thus, by taking a and b sufficiently small, the extent of the light sheet can be made large compared to the area of Σ , and once more we expect that for sufficiently small b , the entropy contained on the light sheet, $S(L)$, will become too great to be bounded by $A/4$.

As a next step, we consider a general (no longer assumed to be conformally flat) space-time, in which the entropy is modelled by a density bounded away from zero and non-decreasing along any future directed timelike curve. Once more, by restricting our attention to a sufficiently small region of space-time, we can assume that the deviation from a flat metric is arbitrarily small, and the conclusion will once again follow.

Thus we can conclude that in the case where entropy is modelled as a density on some fixed background geometry, the FSB bound is violated.

Each of these cases is suggestive, but in order to argue that the conjecture itself fails, the situation must be adapted to a full solution of the Einstein equations. To this end, we should note that the above argument takes place in a homogeneous background, as does much of the discussion of Bekenstein, Fischler and Susskind, and Bousso, cited above. However, in a real space-time there are fluctuations and inhomogeneities, whose presence must be acknowledged, and indeed these inhomogeneities are essential to the discussion of entropy.

Note that the effect of such fluctuations provides the basis of the arguments of Tavakol and Ellis [9] that the light sheet should be truncated at self-intersections. Their argument, though, is based on the observation that truncating only at conjugate points yields a surface of extremely complicated geometry, rendering the light-sheet effectively uncomputable. They suggest truncating earlier as a cure for this problem of operational definability. It is the intent of the current paper to strengthen their arguments by arguing that the covariant entropy conjecture is in fact violated, not just difficult to work with.

Fortunately, the presence of such fluctuations does not destroy the argument presented in the previous section. For if the size of the fluctuations in the energy density of the matter fields is small compared with the energy density, then we expect only a small proportional change in the distance along each null geodesic to the first point where the expansion vanishes. (Even though the locus defined by such points becomes much more complicated.)

We thus expect that just as in the case where entropy is modelled as a density on an averaged background, the amount of entropy on the light sheet will become large compared to the area of our small spacelike ellipsoid, if it is sufficiently oblate. Therefore we expect the covariant entropy conjecture to be false in general for such sufficiently small oblate ellipsoids. We conclude from that above considerations that the light sheet used in the holography bound is too large, and that a weakened form of the covariant entropy conjecture would be more useful.

3 Discussion

We should now address the issue of just how this suggests that we weaken the covariant entropy conjecture. A motivation is provided by a standard result from causal theory.

Suppose, then, that Σ is a spacelike two-surface lying in a Cauchy surface, \mathcal{S} , of an open globally hyperbolic space-time \mathcal{M} (so that the notion of inside Σ is unambiguous), such that the inward directed, orthogonal null congruence \mathcal{C} composed of null geodesics proceeding to the future has negative expansion everywhere. Let L be the light sheet associated with \mathcal{C} , and let B denote a spacelike three-surface with boundary Σ . (B is not assumed to lie in the Cauchy surface mentioned above.)

Now, consider a point $p \in L$ which lies to the future of a point of self-intersection, s , of \mathcal{C} . As Bousso and Tavakol and Ellis observe [5, 9] this may lead to a multiple counting of some of the entropy passing through B ; thus $S(L)$ gives an upper bound for the entropy of any spacelike three-surface spanning Σ . But the situation is rather worse than this observation alone indicates.

For q is connected to s by a future-directed null geodesic segment, γ_1 , and s is connected to p by a future-directed null geodesic segment γ_2 , where γ_1 and γ_2 are not segments of a single null geodesic. It follows [10] that $q \in I^-(p)$. But since $I^-(p)$ is open, and $p \in \partial B$, it follows that there are points in \mathcal{S} outside B to the chronological past of p .

This implies that there are points outside $D(B)$, the Cauchy development of B , which lie to the past of p . As a consequence, entropy from regions which cannot be regarded as inside Σ can contribute to the entropy on the light sheet. Indeed, in the case of the (nearly) flattened ellipsoid described above, most of the entropy on the light sheet comes from a large region outside Σ . Thus entropy passing through L may greatly exceed that contained in any spacelike three-surface spanning Σ .

This provides a physical interpretation for the violation of the FSB bound, and strongly suggests that we should also stop the light sheet at points of self-intersection, thereby ensuring that inappropriate entropy does not contribute. Denote this truncated light sheet by L' .

In the case where the future horismos of B , $E^+(B)$, is composed of null geodesic segments starting on Σ , and this congruence has negative expansion everywhere on Σ , $E^+(B)$ coincides with L' . We then see that in this case the entropy passing through the light sheet all starts off at some point inside Σ , and there is no contribution from regions outside Σ . Thus in this case, the entropy contained on the (truncated) light sheet, $S(L')$ satisfies $S(L) \geq S(L') \geq S(B)$. The first inequality follows from the fact that $L' \subseteq L$, and the second from the second law of thermodynamics. Indeed, $S(L')$ is the least upper bound of $S(B)$ where B ranges over all three-surfaces spanning Σ . This seems to be a suitable choice of light sheet, and again allows the recovery of the Bekenstein bound in the relevant situations.

4 Conclusion

We have argued that the covariant entropy conjecture is violated for sufficiently oblate small ellipsoidal surfaces, and provided a physical interpretation for the source of the violation. This suggests a particular adaptation of the light sheet in which we do not include points beyond self-intersections of the null congruence. Finally, we note again that this does not attempt to invalidate the holographic entropy programme; it rather argues that some of the inequalities used so far need to be replaced by rather weaker ones.

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